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LOW RAYLEIGH NUMBER HEAT TRANSFER IN DUCTS OF VARYING CROSS-SECTION

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NOMENCLATURE

a,	radius of the tube cross-section;
a_0,	mean radius of the tube cross-section;
a_n, b_n, c_n,	coefficients appearing in the solution and listed in the Appendix;
g,	acceleration due to gravity;
k,	thermal diffusivity of the fluid;
Nu,	Nusselt number, $-(\nu/k)\left.\frac{\partial\theta}{\partial\eta}\right\vert_{\eta=1}$;
P, p,	dimensional and non-dimensional pressures, respectively;
Pr,	Prandtl number, ν/k;
Ra,	Rayleigh number, $\tau g\beta a_0^4/\nu k$;
Re,	Reynolds number, $\psi_0/a_0\nu$;
s,	slowly varying function of the axial coordinate;
s',	ds/dx;
s'',	d^2s/dx^2;
T,	dimensional temperature;
T_w,	specified wall temperature;
T_0,	temperature at the cross-section containing the origin;
(R, ϕ, X), (r, ϕ, x),	dimensional and non-dimensional cylindrical polar coordinates;
(U, V, W), (u, v, w),	velocity components in the directions (R, ϕ, X) and (r, ϕ, x), respectively.

Greek symbols

β,	coefficient of cubical expansion of the fluid;
ε,	a small parameter signifying the slow variation of the tube cross-section, $\varepsilon \ll 1$;
η,	r/s;
θ,	non-dimensional temperature;
ν,	coefficient of kinematic viscosity of the fluid;
π,	modified stream function;
ρ,	density of the fluid;
τ,	constant temperature gradient at which the tube wall is maintained;
ψ_0,	stream constant at the tube wall.

1. INTRODUCTION

IN VIEW of the fact that the secondary motion arising due to buoyancy forces in the flow field of a viscous fluid significantly increases the rate of heat transfer with increasing axial distance, the natural convective effects have been extensively analysed in recent years [1–9]. However, most of the work deals with experimental models while the theoretical models are limited in scope due to the complicated equations of motion and energy. Though closed form solutions have been presented in the case of a straight circular tube [7–9], the axial dependence of the flow variables has been neglected. The aim of the present paper is to obtain analytical expressions for the free convective flow and temperature distributions as a next approximation over the forced convection solution, in ducts of non-uniform cross-section which are encountered in heat exchangers and in biological systems.

2. FORMULATION OF THE PROBLEM

We consider the fully developed laminar steady motion of a viscous incompressible fluid in a horizontal tube of radius a with non-uniform cross-section, the walls of which are heated uniformly so that a constant temperature gradient τ is maintained along the axial direction. Cylindrical polar coordinates (R, ϕ, X) are used with ϕ measured anticlockwise from the upward vertical and X along the axis of the tube; the corresponding velocities are (U, V, W). We neglect the density variations insofar as they give rise to a gravitational force and neglect the temperature dependence of the coefficient of kinematic viscosity ν and the thermometric conductivity k of the fluid. These assumptions are well justified since the wall temperature increases slowly with distance along the tube axis. In view of the smaller velocities and comparatively larger temperature differences involved we can also neglect the dissipation terms in the energy equation.

The non-uniformity of the tube cross-section is introduced based on the long wavelength approximation. This is achieved by representing the radius of the tube ($R = a$) as a slowly varying function of the axial coordinate

$$a = a_0 s(\varepsilon X/a_0) \tag{2.1}$$

where a_0 is the mean radius of the tube, ε is a small parameter ($\varepsilon \ll 1$) signifying the small variation.

The basic equations are rendered dimensionless using the following scheme:

$$\begin{aligned} r &= R/a_0, \\ x &= \varepsilon X/a_0, \\ (u, v, w) &= (a_0^2/\psi_0)(U, V, W), \\ p &= (a_0^4/\rho\psi_0^2)P, \\ T_w - T &= (\tau a_0 \nu/k)\theta \end{aligned} \tag{2.2}$$

where T_w the specified wall temperature is given by

$$T_w = T_0 + \tau X. \tag{2.3}$$

Thus the governing equations are

$$\frac{\partial(ru)}{\partial r} + \frac{\partial v}{\partial \phi} + \frac{\partial(rw)}{\partial x} = 0 \tag{2.4}$$

$$u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \phi} - \frac{v^2}{r} + \varepsilon w\frac{\partial u}{\partial x} = -\frac{\partial p}{\partial r} + \frac{1}{Re}\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \phi}\right] - (Ra/Re^2)\theta\cos\phi \tag{2.5}$$

$$u\frac{\partial v}{\partial r}+\frac{v}{r}\frac{\partial v}{\partial \phi}+\frac{uv}{r}+\varepsilon w\frac{\partial v}{\partial x}=-\frac{1}{r}\frac{\partial p}{\partial \phi}$$
$$+\frac{1}{Re}\left[\nabla^2 v-\frac{v}{r^2}+\frac{2}{r^2}\frac{\partial u}{\partial \phi}\right]+(Ra/Re^2)\theta\sin\phi \quad (2.6)$$

$$u\frac{\partial w}{\partial r}+\frac{v}{r}\frac{\partial w}{\partial \phi}+\varepsilon w\frac{\partial w}{\partial x}=-\frac{\partial p}{\partial x}+\frac{1}{Re}\nabla^2 w \quad (2.7)$$

and

$$u\frac{\partial \theta}{\partial r}+\frac{v}{r}\frac{\partial \theta}{\partial \phi}+w\left(\varepsilon\frac{\partial \theta}{\partial x}-\frac{1}{Pr}\right)=\frac{1}{Pr\,Re}\nabla^2\theta \quad (2.8)$$

where

$$\nabla^2\equiv\frac{\partial^2}{\partial r^2}+\frac{1}{r}\frac{\partial}{\partial r}+\frac{1}{r^2}\frac{\partial^2}{\partial \phi^2}+\varepsilon^2\frac{\partial^2}{\partial x^2}.$$

3. SOLUTIONS

We shall take the solution of equations (2.4)–(2.8) to consist of two parts, one due to the variation of the tube cross-section and the other due to buoyancy effect. Accordingly a suitable perturbation scheme for the solution would be

$$Y(r,\phi,x)=Y_{00}+(\varepsilon Y_{10}+RaY_{11})+(\varepsilon^2Y_{20}+\varepsilon RaY_{21}+Ra^2Y_{22})+\ldots \quad (3.1)$$

where Y_{nn} takes the variables u, v, w, p and θ. Substitution of equation (3.1) into equations (2.4)–(2.8) and separating the coefficients of equal powers of the parameters yield a number of equations. These equations can be grouped again into three parts namely (i) those that govern the momentum and heat transfer in tubes of varying cross-section, (ii) those that yield the free convection flow and (iii) those that give the interaction of the two factors.

With such separating effects we notice that the equations and solutions for (i) is same as that obtained by Manton [10], while the corresponding temperature distribution is given by

$$\theta_{00}=\frac{Re}{4}(1-\eta^2)(3-\eta^2),$$

$$\theta_{10}=(Re^2s'/72s)(\eta^8-8\eta^6+18\eta^4-16\eta^2+5),$$

$$\theta_{20}=Re^3=\left\{\sum_{n=1}^{6}\frac{(1-\eta^{2n})}{2n}\left[(-1)^{n+1}a_n+\frac{Pr}{144}\frac{b_n}{n}\right]\right\}+\frac{Re}{6}\sum_{n=1}^{3}(-1)^{n+1}c_n(1-\eta^{2n}) \quad (3.2)$$

where $\eta=r/s$ and for convenience the constants a_n, b_n and c_n are recorded in the Appendix. In case (ii) the equations of motion and energy and their solutions will be the same as that obtained by Morton [7] except for fractional multiplicative changes arising due to the definitions of the Rayleigh number which differ by a multiplication factor of 4.

Thus case (iii) gives an improved extension to the analyses of Manton [10] and Morton [7]. The evaluation of these equations provides the results of the more important contribution due to the combined effect of the non-uniformity of the tube cross-section and the presence of a body force. The governing equations in this case are

$$\frac{\partial}{\partial r}(ru_{21})+\frac{\partial v_{21}}{\partial \phi}+\frac{\partial(rw_{11})}{\partial x}=0, \quad (3.3)$$

$$u_{10}\frac{\partial u_{11}}{\partial r}+u_{11}\frac{\partial u_{10}}{\partial r}+w_{00}\frac{\partial u_{11}}{\partial x}=-\frac{\partial p_{21}}{\partial r}+\frac{1}{Re}\left[\nabla^2u_{21}-\frac{u_{21}}{r^2}-\frac{2}{r^2}\frac{\partial v_{21}}{\partial \phi}\right]-\frac{1}{Re^2}\theta_{10}\cos\phi, \quad (3.4)$$

$$u_{10}\frac{\partial v_{11}}{\partial r}+\frac{u_{10}v_{11}}{r}+w_{00}\frac{\partial v_{11}}{\partial x}=-\frac{1}{r}\frac{\partial p_{21}}{\partial \phi}+\frac{1}{Re}\left[\nabla^2v_{21}-\frac{v_{21}}{r^2}+\frac{2}{r^2}\frac{\partial u_{21}}{\partial \phi}\right]+\frac{1}{Re^2}\theta_{10}\sin\theta, \quad (3.5)$$

$$u_{10}\frac{\partial w_{11}}{\partial r}+u_{11}\frac{\partial w_{10}}{\partial r}+u_{21}\frac{\partial w_{00}}{\partial r}+\frac{\partial}{\partial x}(w_{00}w_{11})=-\frac{\partial p_{11}}{\partial x}+\frac{1}{Re}\nabla^2w_{21}, \quad (3.6)$$

$$u_{10}\frac{\partial \theta_{11}}{\partial r}+u_{11}\frac{\partial \theta_{10}}{\partial r}+u_{21}\frac{\partial \theta_{00}}{\partial r}+w_{00}\frac{\partial \theta_{11}}{\partial x}+w_{11}\frac{\partial \theta_{00}}{\partial x}-\frac{1}{Pr}w_{21}=\frac{1}{Pr\,Re}\nabla^2\theta_{21}$$

To solve these equations we introduce the modified stream function π using equation (3.3) as follows:

$$u_{21}=\frac{1}{r}\frac{\partial \pi}{\partial \phi}\quad\text{and}\quad \underline{V}_{21}=-\frac{\partial \pi}{\partial r}$$

where

$$V_{21}=v_{21}+\frac{Re}{1440}ss'\eta^4(\eta^6-15\eta^4+35\eta^2-25)\sin\phi \quad (3.8)$$

and the corresponding boundary conditions are

$$\left.\begin{aligned}&\frac{1}{r}\frac{\partial \pi}{\partial \phi}=0\\&\frac{\partial \pi}{\partial r}=\frac{Re}{360}ss'\sin\phi\\&w_{21}=0,\ \theta_{21}=0\end{aligned}\right\}\quad\text{on}\quad r=s(x) \quad (3.9)$$

and $(1/r)\,(\partial\pi/\partial\phi)$, $(\partial\pi/\partial r)$, w_{21} and θ_{21} are finite on $r=0$. Substituting equations (3.8) into equations (3.4)–(3.7) and using equations (3.9), we get the solution as

$$\pi=-\frac{Res^2s'\sin\phi}{207360}\eta\{s^2[0.7074\eta^{12}+14.175\eta^{10}+45.9378\eta^8-34.3746\eta^6-74.9997\eta^4-5.7474\eta^2+54.3015]+s[8.8336\eta^{11}-128.3696\eta^9+46.3024\eta^7-70.704\eta^5+543.78\eta^2-399.8424]+288(1-\eta^2)\},$$

$$w_{21}=\cos2\phi\{s^2s'\eta(0.0045\eta^6-0.0324\eta^4+0.0783\eta^2-0.0504)+Res'\eta(5.3333\eta^4-15.9999\eta^2+10.6666)+Re^2s'[0.2857\eta^{13}-0.7005\eta^{11}+36.0025\eta^9-85.0056\eta^7+99.3396\eta^5-49.003\eta^3-0.9187]+Re^2ss'\eta[s(0.0001\eta^6+0.0006\eta^4-0.0002\eta^2+0.0001)-(0.0008\eta^4-0.0019\eta^2+0.0011)]\},$$

$$\theta_{21}=\eta\cos\phi\{Pr\,Re^3s^6s'[0.0001\eta^{14}-0.0001\eta^{12}+0.001\eta^{10}-0.001\eta^8+0.002\eta^6-0.002\eta^4+0.0001\eta^2-0.001]-Pr\,Re^2s^2s'(0.0001\eta^6-0.0002\eta^4+0.0004\eta^2-0.0002)+Re^2s'(0.0137\eta^{14}-0.0042\eta^{12}+0.3\eta^{10}-1.0626\eta^8+2.1807\eta^6-2.7085\eta^4+1.2185\eta^2+0.0624)+Res^2s'(0.0001\eta^8-0.0007\eta^6+0.003\eta^4-0.0025)+\text{terms of order }10^{-7}\text{ and less}\},$$

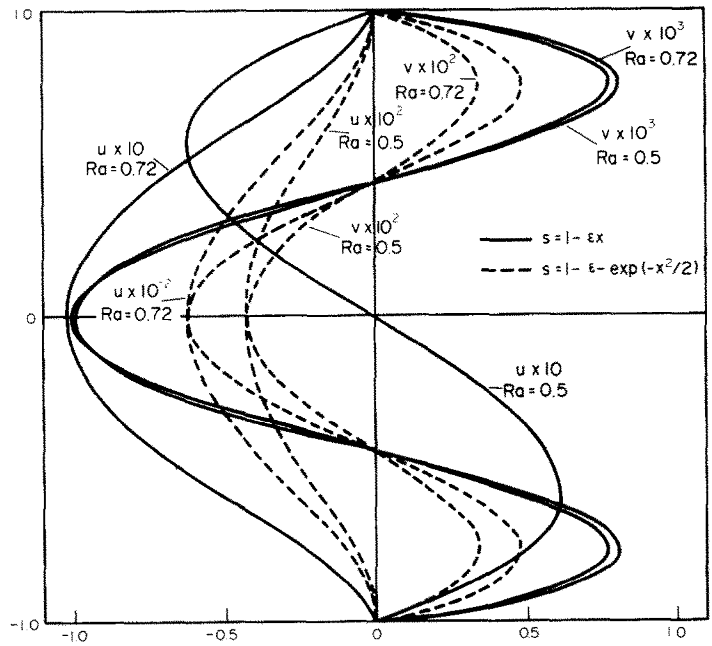

FIG. 1. The non-dimensional radial and tangential velocity components taken up to second order approximation in planes normal to the pipe axis. The radial component u is taken vertically through the pipe axis. The transverse component v is a horizontal profile through the axis.

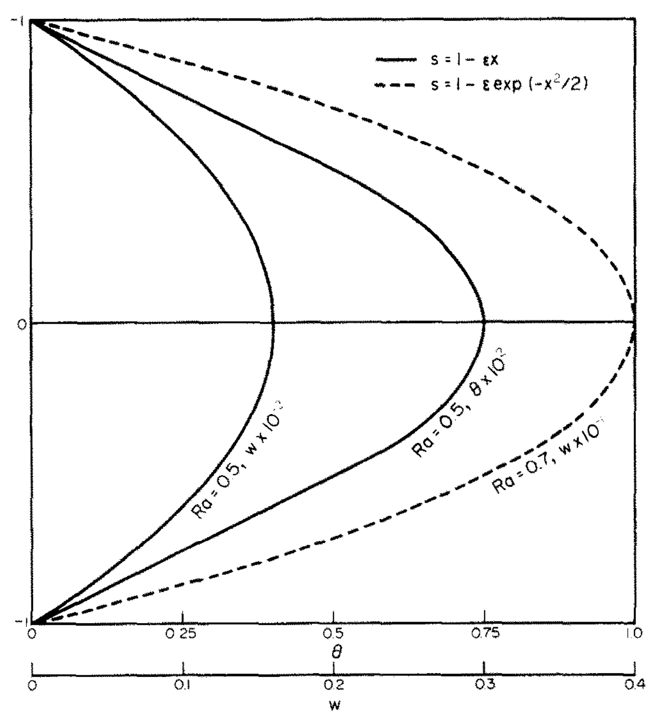

FIG. 2. The non-dimensional axial velocity component and temperature distribution for two different geometries.

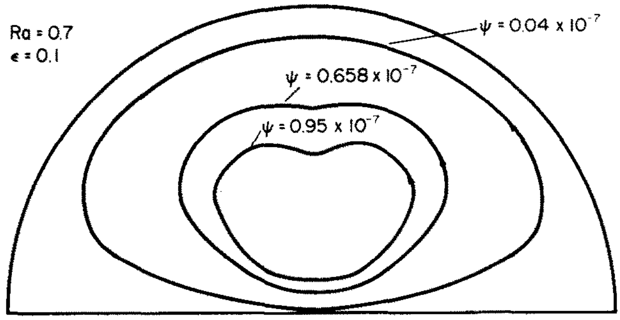

FIG. 3. The stream lines corresponding up to the second order approximation for flow in planes normal to the pipe axis.

the dimensionless heat flux at the wall is given by

$$Nu = -\frac{Pr}{s}\frac{d\theta}{d\eta}\bigg|_{\eta=1} \tag{3.10}$$

and the pressure distribution up to the first order of approximation is given by

$$p = -16\left\{\int_0^x \frac{dx}{\varepsilon s^4} + \frac{Re}{4s^4} + \frac{11}{135}\varepsilon Re^2 \int_0^x s'^2\frac{dx}{s^4} + \frac{11}{180}\varepsilon Re^2\frac{s'}{s^5}\right.$$
$$+\frac{2}{3}\varepsilon\int_0^x (5s'^2 - s'')\frac{dx}{s^4} + \varepsilon\frac{s'}{s^3}\left[\frac{1}{2} + \eta^2\right]$$
$$\left. + \frac{Ra}{788\,Re}(2\eta^5 - 12\eta^3 + 29\eta) \times \int_0^x s\,dx\right\}.$$

4. DISCUSSION

To understand the combined effect of the non-uniformity of the tube cross-section and the secondary convection we have considered flow in tapered tubes i.e. (i) $s(x) = 1 - \varepsilon x$ and flow in constricted tubes i.e. (ii) $s(x) = 1 + \varepsilon \sin 2\pi x$ and (iii) $s(x) = 1 - \varepsilon \exp(-x^2/2)$. Numerical computations are carried out and the results are depicted graphically.

The radial and transverse components of the flows at each cross-section are in the negative direction and only near the wall these become positive. This feature corresponds to the trapping of the fluid observed by Manton [12]. Due to the presence of free convection, however small, there is a significant heating of the fluid only near the wall. This, in turn, enhances the secondary motion resulting in the increase of the velocity near the wall. This phenomenon is gradually attenuated by sufficiently strong convective force. Figure 1 shows that the radial and transverse velocity profiles are similar to that of Morton [7] and hence the magnitude variations can be attributed mainly to the tube geometry.

Figure 2 depicts the axial velocity component and the temperature profiles. The temperature field's dependence on the tube geometry is insignificant as it changes in magnitude only with respect to the free convective force. Whereas, the magnitude of the axial velocity shows a marked dependence both on the tube geometry and on the convection parameter. Figure 3 depicts the deviation of the stream function from that of a rigid tube [7].

Figure 4 shows the variation of the Nusselt number with respect to the tube geometry and for different Rayleigh numbers. The heat flux at the wall prominently fluctuates from its mean. But the amplitude of oscillation decreases as Ra increases and it almost coincides with its mean for sufficiently large Ra. This is to be expected as, in the fully developed region, the secondary flow gradually increases.

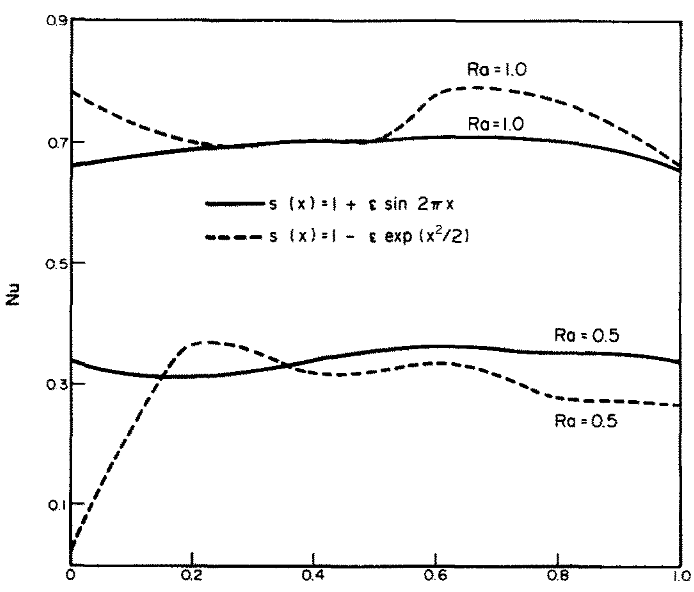

FIG. 4. The Nusselt number as a function of the axial coordinate x.

This minimizes the temperature fluctuations in the flow field by convection. Hence after sufficiently long time, with strong convection, the heat flux becomes steady irrespective of the tube geometry, whereas in the analysis of Chow and Soda [11] it remains oscillatory throughout but the inherent restriction on the value of the Reynolds number in this analysis prevents us from making any further quantitative comparison with their analysis.

It is interesting to note that the pressure distribution is non-uniform along the axial direction as emphasised by Casal and Gill [9] and also that the flow and temperature fields' dependence upon the axial distance cannot be neglected.

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APPENDIX

The coefficients appearing in the solutions of the governing equations are

$$a_1 = \frac{1}{900}\left(\frac{19}{24}P_1 + 13Q_1\right) \qquad b_1 = -(1.75P_1 - Q_1)$$

$$a_2 = \frac{1}{720}\left(\frac{11}{3}P_1 + 29Q_1\right) \qquad b_2 = 13.75P_1 - 7Q_1$$

$$a_3 = \frac{1}{24}\left(\frac{2}{9}P_1 + Q_1\right) \qquad b_3 = -(35.5P_1 + 16Q_1)$$

$$a_4 = \frac{1}{144}(P_1 + 3Q_1) \qquad b_4 = 34.5P_1 + 10Q_1$$

$$a_5 = \frac{1}{480}\left(P_1 + \frac{8}{3}Q_1\right) \qquad b_5 = -(10.75P_1 - 5Q_1)$$

$$a_6 = \frac{1}{1800}\left(\frac{1}{3}P_1 + Q_1\right) \qquad b_6 = -\left(\frac{1}{8}P_1 + Q_1\right)$$

where

$$P_1 = 16(s'/s)^2 \quad \text{and} \quad Q_1 = 1.25P_1 - 4(s''/s)$$

$$c_1 = 2c_3 = 5s'^2 - ss'', \; c_2 = 7.25s'^2 - 1.75ss''.$$